

9/20

ex) Reparameterize  $\vec{r}(t) = \langle 3\sin(t), 2t, 3\cos(t) \rangle$  by arc length  
- measure from  $t=0$

Sol<sup>n</sup>) First we compute the arc length function.

$$S(t) = \int_{t=0}^t |\vec{r}'(a)| da$$

$$\begin{aligned}\vec{r}'(t) &= \langle 3\cos(t), 2, -3\sin(t) \rangle \Rightarrow |\vec{r}'(t)| = \sqrt{(3\cos(t))^2 + 2^2 + 3^2\sin^2(t)} \\ &= \sqrt{3^2 + 2^2} = \sqrt{13} \Rightarrow S(t) = \int_{a=0}^t \sqrt{13} da = [\sqrt{13}a]_0^t = \sqrt{13}(t-0)\end{aligned}$$

$$\therefore S(t) = \sqrt{13} t$$

- Thus the time is determined by:  $t = \frac{S}{\sqrt{13}}$   
- Now we replace the parameter  $t$ :

$$\vec{P}(s) = \vec{r}(t(s)) = \vec{r}\left(\frac{s}{\sqrt{13}}\right) = \left\langle 3\sin\left(\frac{s}{\sqrt{13}}\right), \frac{2}{\sqrt{13}}s, 3\cos\left(\frac{s}{\sqrt{13}}\right) \right\rangle$$

NB: for  $\vec{r}(t)$  as above,

$$\vec{P}(s) = \left\langle \frac{3}{\sqrt{13}}\cos\left(\frac{s}{\sqrt{13}}\right), \frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}\sin\left(\frac{s}{\sqrt{13}}\right) \right\rangle$$

So,

$$\begin{aligned}|\vec{P}(s)| &= \sqrt{\left(\frac{3}{\sqrt{13}}\right)^2 \cos^2\left(\frac{s}{\sqrt{13}}\right) + \left(\frac{2}{\sqrt{13}}\right)^2 + \left(\frac{3}{\sqrt{13}}\right)^2 \sin^2\left(\frac{s}{\sqrt{13}}\right)} \\ &= \sqrt{\frac{9}{13} + \frac{4}{13}} = \sqrt{\frac{13}{13}} = 1\end{aligned}$$

So,  $\vec{P}(s)$  is a unit-speed parameterization

### Physicsy nonsense

ex) Find the velocity and acceleration of the curve...

$$\vec{r}(t) = \langle 2^t, t^2, \ln(t+1) \rangle \text{ at time } t=1$$

Sol<sup>n</sup>:  $\vec{v}(t) = \vec{r}'(t) = \langle \ln(2)e^{2t}, 2t, (t+1)^{-1} \rangle$

$$= \langle \ln(2)2^t, 2t, (t+1)^{-1} \rangle$$

$$\vec{v}(1) = \langle \ln(2)2^1, 2 \cdot 1, (1+1)^{-1} \rangle = \langle 2\ln(2), 2, \frac{1}{2} \rangle$$

$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t) = \langle (\ln(2))^2 2^t, 2, -(t+1)^{-2} \rangle = \langle \ln(2)^2 2^t, 2, -(t+1)^{-2} \rangle$

$$\vec{a}(1) = \langle 2\ln(2)^2, 2, -\frac{1}{4} \rangle$$

ex) Find velocity and position of the curve satisfying

$$\vec{a}(t) = \langle 2, 0, 2t \rangle, \quad \vec{v}(0) = \langle 3, -1, 0 \rangle, \quad \vec{r}(0) = \langle 1, 0, 1 \rangle$$

Sol<sup>n</sup>:  $\vec{v}(t) = \int \vec{a}(t) dt = \langle 2t, 0, t^2 \rangle + \vec{C}$

$$\Rightarrow \langle 3, -1, 0 \rangle = \vec{v}(0) = \langle 2(0), 0, (0)^2 \rangle + \vec{C}$$

$$\therefore \vec{C} = \langle 3, -1, 0 \rangle - \langle 0, 0, 0 \rangle = \langle 3, -1, 0 \rangle$$

$$\therefore \vec{v}(t) = \langle 2t+3, -1, t^2 \rangle$$

so,  $\vec{r}(t) = \int \vec{v}(t) dt = \langle t^2+3t, -t, \frac{t^3}{3} \rangle + \vec{d}$

$$\Rightarrow \langle 1, 0, 1 \rangle = \vec{r}(0) = \langle 0^2+3(0), -(0), \frac{0^3}{3} \rangle + \vec{d} = \vec{d}$$

$$\vec{r}(t) = \langle t^2+3t, -t, \frac{1}{3}t^3 \rangle + \vec{d}$$

$$\vec{r}(t) = \langle t^2+3t+1, -t, \frac{1}{3}t^3+1 \rangle$$

←

ex) When is the particle w/ position function  $\vec{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$  moving the slowest?

Sol<sup>n</sup>: Want to minimize the speed of  $\vec{r}(t)$   
i.e. Want to compute minimum of  $f(t) = |\vec{r}'(t)|$

$\vec{r}(t) = \langle 2t, 5, 2t - 16 \rangle$ , so  
 $f(t) = |\vec{r}'(t)| = \sqrt{(2t)^2 + 5^2 + (2t - 16)^2} = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256} = \sqrt{8t^2 - 64t + 281}$

→ First derivative test:

$$f'(t) = \frac{1}{2}(8t^2 - 64t + 281)^{-\frac{1}{2}}(16t - 64)$$

$$= \frac{8t - 32}{\sqrt{8t^2 - 64t + 281}}$$

Note:  $(-64)^2 - 4 \cdot 8 \cdot 281 = 2^{12} - 2^5 \cdot 281 < 2^{12} - 2^5 \cdot 256$   
 $= 2^{12} - 2^5 \cdot 2^8 = 2^{12} - 2^{13} < 0$

∴  $8t^2 - 64t + 281 \neq 0$  for all  $t$

∴ the only critical points for  $f$  are  $8t - 32 = 0$ , i.e.  $t = 4$

Now,  $f'(0) = \frac{-32}{\sqrt{281}} < 0$  and  $f'(6) = \frac{8}{\sqrt{7}} > 0$

∴ the ~~slow~~ particle is slowest at  $t = 4$ . U

alternative sol<sup>n</sup>: if  $f(t) > 0$  for all  $t$ , then the critical points of  $(f(t))^2 = g(t)$  are precisely those of  $f(t)$ .

As before,  $f(t) = |\vec{r}'(t)| = \sqrt{8t^2 - 64t + 281}$

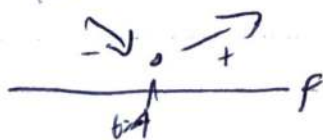
So,  $f(t) > 0$  for all  $t$  b/c  $(2t)^2 + 5^2 + (2t - 16)^2 \geq 5^2$  for all  $t$

∴ we can minimize  $g(t) = (f(t))^2 = 8t^2 - 64t + 281$

→  $g'(t) = 0$  iff  $16t - 64 = 0$  iff  $t = 4$

first derivative test

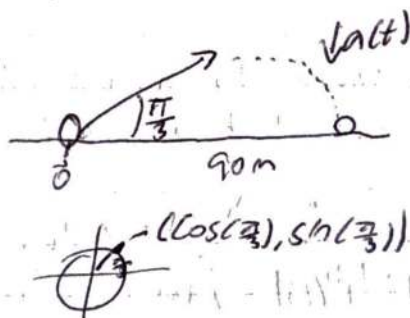
yields a minimum at  $t = 4$



ex) A ball is kicked from the ground at an angle of  $60^\circ$  if the ball lands 90m away, what was the initial speed of the ball?  
 $g = 9.8 \text{ m/s}^2$  towards earth

Sol<sup>n</sup>:

$$\begin{aligned} \vec{r}(0) &= \langle 0, 0 \rangle \\ \vec{a}(t) &= \langle 0, -9.8 \rangle = \langle 0, -\frac{49}{5} \rangle \\ \vec{v}(0) &= |\vec{v}(0)| \langle \cos(\frac{\pi}{3}), \sin(\frac{\pi}{3}) \rangle \end{aligned}$$



$$\vec{v}(0) = |\vec{v}(0)| \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle, \quad \vec{r}(0) = \langle 90, 0 \rangle$$

Want:  $C = |\vec{v}(0)|$

Now,  $\vec{v}(t) = \int \vec{a}(t) dt = \langle \alpha, -\frac{49}{5}t + \beta \rangle$

We have,  $C \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = \vec{v}(0) = \langle \alpha, -\frac{49}{5}t + \beta \rangle$

$$\alpha = \frac{C}{2}, \quad \beta = \frac{\sqrt{3}}{2}C$$

$$\therefore \vec{v}(t) = \langle \frac{1}{2}C, -\frac{49}{5}t + \frac{\sqrt{3}}{2}C \rangle$$

$$\therefore \vec{r}(t) = \int \vec{v}(t) dt = \langle \frac{1}{2}Ct + 8, -\frac{49}{10}t^2 + \frac{\sqrt{3}}{2}Ct + 8 \rangle$$

Now  $\langle 0, 0 \rangle = \vec{r}(0) = \langle \frac{1}{2}C(0) + 8, -\frac{49}{10}(0)^2 + \frac{\sqrt{3}}{2}C(0) + 8 \rangle = \langle 8, 8 \rangle$

$$\vec{r}(t) = \langle \frac{1}{2}Ct, -\frac{49}{10}t^2 + \frac{\sqrt{3}}{2}Ct \rangle$$

So at time  $t = b$ :  $\begin{cases} \frac{1}{2}Cb = 90 \\ -\frac{49}{10}b^2 + \frac{\sqrt{3}}{2}Cb = 0 \end{cases} \therefore \begin{cases} -\frac{49}{10}b^2 + 90\sqrt{3} = 0 \\ C = \frac{180}{b} \end{cases}$

$$\therefore \frac{900\sqrt{3}}{49} = b^2 \Rightarrow b = \pm \frac{30(3^{1/4})}{7}, \text{ reject negative}$$

$$\text{So } C = \frac{180}{\frac{30 \cdot 3^{1/4}}{7}} = \frac{42}{3^{1/4}}$$